# Talk to the CALF-seminar, Oxford 23.6.2005

## Reflection Groups, Generalised Cartan Matrices & Kac-Moody Algebras

Kyriakos Papadopoulos
kyriakos_hellas@yahoo.gr

# Foreword

This talk will be an introduction to *infinite dimensional (or Kac-Moody) Lie algebras*. The theory for these algebras was developed in the late sixties by Kac (while trying to understand and generalise works of Guillemin-Quillen-Singer-Sternberg-Weisfeiler on Cartan's classification) and Moody (when independently undertook the study of the Lie algebas $\mathfrak{g}'(A)$).

These algebras are defined by *Generalised Cartan Matrices*, so we find it important to introduce the theory of these matrices, and we also find it vital to talk about *Reflection Groups of Integral Hyperbolic Lattices*, since all the theory that we will establish in this talk will depend on these type of groups.

The author would like to thank prof. V.V. Nikulin for giving him the opportunity to study this subject.

# Contents



## 0.1 Reflection Groups of Integral Hyperbolic Lattices and Generalised Cartan Matrices[1]

In this section we will give all the basic definitions on generalised Cartan matrices, and show how they are related to reflection groups of integral hyperbolic lattices.

**Definition 1** For a countable set of indices $I$, a finite-rank matrix $A = (a_{ij})$ is called *generalised Cartan matrix*, if and only if:

(i) $a_{ii} = 2\,;$

(ii) $a_{ij} \in \mathbb{Z}^-$, $i \neq j$ and

(iii) $a_{ij} = 0 \Rightarrow a_{ji} = 0\,.$

We consider such a matrix to be *indecomposable*, i.e. there does not exist a decomposition $I = I_1 \cup I_2$, such that $I_1 \neq \emptyset$, $I_2 \neq \emptyset$ and $a_{ij} = 0$, for $i \in I_1$, $j \in I_2$.

[1]The presentation of this section is based on [1], [2]and [3].

**Definition 2** A generalised Cartan matrix $A$ is *symmetrizable*, if there exists invertible diagonal matrix $D = \operatorname{diag}(\ldots \epsilon_i \ldots)$ and symmetric matrix $B = (b_{ij})$, such that:

$$A = DB \text{ or } (a_{ij}) = (\epsilon_i \, b_{ij}),$$

where $\epsilon_i \in \mathbb{Q}$, $\epsilon_i > 0$, $b_{ij} \in \mathbb{Z}^-$, $b_{ii} \in 2\mathbb{Z}$ and $b_{ii} > 0$.

The matrices $D$ and $B$ are defined uniquely, up to a multiplicative constant, and $B$ is called *symmetrised generalised Cartan matrix.*

**Remark 1** A symmetrizable generalised Cartan matrix $A = (a_{ij})$ and its symmetrised generalised Cartan matrix $B = (b_{ij})$ are related as follows:

$$(a_{ij}) = \left(\frac{2\, b_{ij}}{b_{ii}}\right),$$

where $b_{ii} | 2\, b_{ij}$.

**Definition 3** A symmetrizable generalised Cartan matrix $A$ is called *hyperbolic*, if its symmetrised generalised Cartan matrix $B$ has exactly one negative square (or $A$ has exactly one negative eigenvalue).

Let us now recall the definition for *hyperbolic integral quadratic form $S$* :

**Definition 4** A *hyperbolic integral symmetric bilinear form* (or hyperbolic integral quadratic form) on a finite rank free $\mathbb{Z}-$ module $M$, of dimension $n$, over the ring of integers, is a map:

$$S : M \times M \to \mathbb{Z},$$

satisfying the following conditions:

(i) $S(\alpha\, m_1 + \beta\, m_2, m_3) = \alpha\, S(m_1, m_3) + \beta\, S(m_2, m_3)$;

(ii) $S(m_3, \alpha\, m_1 + \beta\, m_2) = \alpha\, S(m_3, m_1) + \beta\, S(m_3, m_2)$;

(iii) $S(m_1, m_2) = S(m_2, m_1)$ (symmetry) and

(iv) signature$=(n, 1)$; i.e. in a suitable basis, the corresponding matrix of $S$ is a diagonal matrix with $n$ positive squares and one negative square in the diagonal,

where $\alpha$, $\beta \in \mathbb{Z}$ and $m_1, m_2, m_3 \in M$.

We now consider an *integral hyperbolic lattice* $(M, S)$, i.e. a pair of a free $\mathbb{Z}-$module $M$ and a hyperbolic integral symmetric bilinear form $S$.

By considering the corresponding cone:

$$V(M) = \{x \in M \otimes \mathbb{R} : (x, x) < 0\},$$

and by choosing its half-cone $V^+(M)$, we can define the corresponding *hyperbolic (or Lobachevskii) space*:

$$\Lambda^+(M) = V^+(M)/\mathbb{R}^{++},$$

as a section (slice) of the cone (a cone is by definition the set of rays with origin at zero), by a hyperplane.

After defining the hyperbolic space, we can work in hyperbolic geometry, by defining the *distance* $\rho$ between two points $\mathbb{R}^{++}x$ and $\mathbb{R}^{++}y$ in $\Lambda^+(M)$, as follows:

$$cosh\,\rho\,(\mathbb{R}^{++}x, \mathbb{R}^{++}y) = \frac{-S(x, y)}{\sqrt{S(x, x)S(y, y)}}$$

Obviously these two points in hyperbolic space are rays in the half-cone $V^+(M)$.

**Remark 2** By definition, when we use signature $(n, 1)$, the square of a vector which is outside the cone $V(M)$ is strictly greater than zero and the square of a vector which is inside the cone is strictly negative. Furthermore, if a vector lies on the surface of the cone, then its square is zero. Obviously, we put a minus sign in the numerator of the definition of distance in hyperbolic space, because the hyperbolic cosine should be always positive.

Now, each element $\alpha_i \in M \otimes \mathbb{R}$, with $(\alpha_i, \alpha_i) > 0$, defines the half spaces:

$$H^+_{\alpha_i} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) \leq 0\},$$

$$H^+_{\alpha_i^-} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) > 0\},$$

which are bounded by the hyperplane:

$$H_{\alpha_i} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) = 0\},$$

where $\alpha_i \in M \otimes \mathbb{R}$ is defined up to multiplication on elements of $\mathbb{R}^{++}$. The hyperplane $H_{\alpha_i}$ is also called *mirror of symmetry.*

Let us denote by $O(M)$ the *group of automorphisms*, which preserves the cone $V(M)$. Its subgroup $O^+(M) \subseteq O(M)$ is of index 2, and fixes the half-cone $V^+(M)$. Furthermore, $O^+(M)$ is discrete in $\Lambda^+(M)$, and has fundamental domain of finite volume.

**Proposition 1** *The index* $[O(M) : O^+(M)]$ *is always equal to* 2.

*Proof.* The group $O(M)$ acts on two elements which are the half cones $V^+$ and its opposite $V^-$. So, the kernel of this action, $O^+(M)$, cannot have index greater than 2. Also, the element $-1$, from $O(M)$, changes $V^+$ with $V^-$. Thus the index is exactly equal to 2. □

**Definition 5** For $(\alpha_i, \alpha_i) > 0$, by $s_{a_i} \in O^+(M)$ we define *reflection* in a hyperplane $H_{\alpha_i}$, of $\Lambda^+(M)$, as follows:

$$s_{\alpha_i}(x) = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i,$$

where $x \in M$ and $\alpha_i \in M$.

**Remark 3** Why this equation, that we gave for reflection in our hyperbolic lattice, works? Answer: because of the following two facts:

(i) for $x = \alpha_i$, we get that $s_{\alpha_i}(\alpha_i) = -\alpha_i$ and

(ii) for $x$ perpendicular to $\alpha_i$, we have that $s_{\alpha_i}(x) = x$,

both of which show that our formula cannot work if we omit number 2 from the numerator.

An obvious remark is that a reflection $s_{a_i}$ changes place between the half-spaces $H^+_{a_i}$ and $H^+_{a_i^-}$.

Going a bit further, if an orthogonal vector $\alpha_i \in M$, $(\alpha_i, \alpha_i) > 0$, in a hyperplane $H_{a_i}$ of $\Lambda^+(M)$, is a *primitive root*, i.e. its coordinates are coprime numbers, then:

$$\frac{2(M, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \subseteq M \Leftrightarrow (\alpha_i, \alpha_i) |\, 2(M, \alpha_i)$$

**Definition 6** Any subgroup of $O(M)$ (the corresponding *discrete* group of motions of $\Lambda(M)$), generated by reflections, is called *reflection group.*

We denote by $W(M)$ the subgroup of $O^+(M)$ generated by *all* reflections of $M$, of elements with positive squares (always for signature $(n, 1)$).

We will also denote by $W$ the subgroup of $W(M)$ generated by reflections in a set of elements of $M$. Obviously, $W \subseteq W(M) \subseteq O^+(M)$ is a subgroup of finite index.

**Definition 7** A lattice $M$ is called *reflective*, if index $[O(M) : W(M)]$ is finite. In other words, $W(M)$ has fundamental polyhedron of finite volume, in $\Lambda(M)$.

Talking a bit more about lattices, we consider again our integral hyperbolic lattice $S : M \times M \rightarrow \mathbb{Z}$. For $m \in \mathbb{Q}$ we denote by $S(m)$ the lattice which one gets, if multiplying $S$ by $m$. If $S$ is reflective, then $S(m)$ is reflective, too.

Furthermore, $S$ is called *even lattice*, if $S(x,x)$ is even, $x \in M$. Otherwise, $S$ is called *odd*.

Last, but not least, $S$ is called *primitive lattice* (or *even primitive*), if $S(\frac{1}{m})$ is not lattice (or even lattice), $m \in \mathbb{N}$, $m \geq 2$.

**Definition 8** A convex polyhedron $\mathcal{M}$, in $\Lambda(M)$, is an intersection:

$$\mathcal{M} = \bigcap_{\alpha_i} H^+_{\alpha_i}$$

of several half-spaces orthogonal to elements $\alpha_i \in M$, $(\alpha_i, \alpha_i) > 0$.

This convex polyhedron is the fundamental chamber for a reflection group $W(M)$, with reflections generated by (primitive) roots, in $M$, each of them orthogonal to exactly one side of this polyhedron. Moreover, we get this chamber if we remove all mirrors of reflection, and take the connencted components of the complement (with the boundary). The fundamental chamber acts *simply transitively*, because if we consider an element $w \in W(M)$, then $w(\mathcal{M}_1) = \mathcal{M}_2$, in other words it fills in our (hyperbolic) space with congruent polyhedra.

The polyhedron $\mathcal{M}$ belongs to the cone:

$$\mathbb{R}^+\mathcal{M} = \{x \in V^+(M) : (x, \alpha_i) \leq 0\},$$

where $\alpha_i \in P(\mathcal{M}) = \{\alpha_i : i \in I\}$; set of orthogonal vectors to $\mathcal{M}$, where exactly one element $\alpha_i$ is orthogonal to each face of $\mathcal{M}$.

We say that $P(\mathcal{M})$ is *acceptable*, if each of its elements is a (primitive) root, which is perpendicular to exactly one side of a convex polyhedron, in $\Lambda^+(M)$.

Also, $\mathcal{M}$ is *non-degenerate*, if it contains a non-empty open subset of $\Lambda^+(M)$ and *elliptic*, if it is a convex envelope of a finite set of points in $\Lambda^+(M)$ or at infinity of $\Lambda^+(M)$.

Let us now return back to the theory for generalised Cartan matrices, and relate it with the material that we introduced for reflection groups of integral hyperbolic lattices.

**Definition 9** An indecomposable hyperbolic generalised Cartan matrix A is equivalent to a triplet:

$$A \sim (M, W, P(\mathcal{M})),$$

where $S : M \times M \to \mathbb{Z}$ is a hyperbolic integral symmetric bilinear form, $W \subseteq W(M) \subseteq O^+(M)$, $W$ is a subgroup of reflections in a set of elements of $M$, with positive squares, $W(M)$ is the subgroup of reflections in all elements of $M$ (with positive squares), $O^+(M)$ is the group of automorphisms, which fixes the half-cone $V^+(M)$, and $P(\mathcal{M}) = \{\alpha_i : i \in I\}$, $(\alpha_i, \alpha_i) > 0$, $A = \left(2\frac{(\alpha',\alpha)}{(\alpha,\alpha)}\right)$, where $\alpha, \alpha' \in P(\mathcal{M})$, $\mathcal{M}$ is a locally finite polyhedron in $\Lambda^+(M)$, $\mathcal{M} = \bigcap_{\alpha_i} H^+_{\alpha_i}$ and $H^+_{\alpha_i} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) \le 0\}$.

The triplet $(M, W, P(\mathcal{M}))$ is called *geometric realisation of* $A$.

Let us now consider $\lambda(\alpha) \in \mathbb{N}$, $\alpha \in P(\mathcal{M})$ and $\gcd(\{\lambda(\alpha) : \alpha \in P(\mathcal{M})\}) = 1$.

$\tilde{\alpha} = \lambda(\alpha)\,\alpha$, $(\tilde{\alpha}, \tilde{\alpha})|\,2(\tilde{\alpha}', \tilde{\alpha}) \Leftrightarrow \lambda(\alpha)(\alpha, \alpha)|\,2\lambda(\alpha')(\alpha', \alpha)$

Then, $\tilde{A} = \Big(2(\tilde{\alpha}', \tilde{\alpha})/(\tilde{\alpha}, \tilde{\alpha})\Big) = \Big(2\lambda(\alpha')(\alpha', \alpha)/\lambda(\alpha)(\alpha, \alpha)\Big)$, $\alpha, \alpha' \in P(\mathcal{M})$, where $\tilde{A}$ is a *twisted, to* A, hyperbolic generalised Cartan matrix and $\lambda(\alpha)$ are called *twisted coefficients of* $\alpha$.

Also, $\tilde{A} = (\tilde{M}, \tilde{W}, \tilde{P}(\tilde{\mathcal{M}})) = (M \supseteq [\{\lambda(\alpha)\,\alpha : \alpha \in P(\mathcal{M})\}\,], W, \{\lambda(\alpha)\,\alpha : \alpha \in P(\mathcal{M})\})$. In other words, $W$ and $\mathcal{M}$ are the same for $\tilde{A}$ and $A$.

Obviously, $A$ is *untwisted*, if it cannot be twisted to any generalised Cartan matrix, different from itself.

Our purpose is to work on hyperbolic generalised Cartan matrices of *elliptic type*, so we need to introduce some more material:

**Definition 10** Let $A$ be a hyperbolic generalised Cartan matrix, $A \sim (M, W, P(\mathcal{M}))$. We define the *group of symmetries of* $A$ (or $P(\mathcal{M})$) as follows:

$$\text{Sym}\ (A) = \text{Sym}\ (P(\mathcal{M})) = \{g \in O^+(M) : g(P(\mathcal{M})) = P(\mathcal{M})\}$$

**Definition 11** A hyperbolic generalised Cartan matrix $A$ has *restricted arithmetic type*, if it is not empty and the semi-direct product of $W$ with $\text{Sym}(A)$, which is equivalent to the semi-direct product of $W$ with $\text{Sym}(P(\mathcal{M}))$, has finite index in $O^+(M)$.

**Remark 4** For $W = (w_1, s_1)$, $S = (w_2, s_2)\,\mathbb{Z}-$modules, the semi-direct product of $W$ with $S$ is equal to $((s_2 w_1) w_2, s_1 s_2)$.

**Definition 12** A hyperbolic generalised Cartan matrix $A$ has *lattice Weyl vector*, if there exists $\rho \in M \otimes \mathbb{Q}$, such that:

$$(\rho, \alpha) = -(\alpha, \alpha)/2,\ \alpha \in P(\mathcal{M})$$

Additionally, $A$ has *generalised lattice Weyl vector*, if there exists $\mathbf{0} \neq \rho \in M \otimes \mathbb{Q}$, such that for constant $N > 0$ :

$$0 \le -(\rho, \alpha) \le N$$

We can think of $\rho$ in $\Lambda^+(M)$ as being the centre of the inscribed circle to $\mathcal{M}$, where $\mathcal{M}$ is the fundamental chamber of a reflection group $W$.

And we can now give the last, and very important, definition of this introductory section:

**Definition 13** A hyperbolic generalised Cartan matrix $A$ has *elliptic type*, if it has restricted arithmetic type and generalised lattice Weyl vector $\rho$, such that $(\rho,\rho) < 0$. In other words, $[O(M) : W] < \infty$ or $\mathrm{vol}(\mathcal{M}) < \infty$ or $P(\mathcal{M}) < \infty$.

## 0.2 The Classification of Generalised Cartan Matrices of Rank 3, of Elliptic Type, with the Lattice Weyl Vector, which are twisted to Symmetric Generalised Cartan Matrices[2]

In this section we are going to describe very briefly how Gritsenko and Nikulin classified in [3] the generalised Cartan matrices of rank 3, of elliptic type (so they have the generalised lattice Weyl vector), which are twisted to symmetric generalised Cartan matrices.

Let $A$ be a generalised Cartan matix of elliptic type, twisted to a symmetric generalised Cartan matrix $\tilde{A}$.This automatically implies that $\tilde{A}$ is of elliptic type, too. Let also $G(A) = (M, W, P(\mathcal{M}))$ be the geometric realisation of $A$, where rank of $A$ is equal to 3.

Furthermore, for $\alpha \in P(\mathcal{M})$, we set:

$$\alpha = \lambda(\alpha)\,\delta(\alpha),$$

where $\lambda(\alpha) \in \mathbb{N}$ are the twisted coefficients of $\alpha$, and $(\delta(\alpha), \delta(\alpha)) = 2$. So, $\tilde{P}(\mathcal{M}) = \{\delta(a) = \alpha/\lambda(\alpha) : \alpha \in P(\mathcal{M})\}$.

Notation: from now on, $\delta(\alpha_i) = \delta_i$ and $\lambda(\alpha_i) = \lambda_i$.

Now, $A$ and its geometric realisation are equivalent to a $(1 + [n/2]) \times n$ matrix $G(A)$ :

$$\text{1st raw: } \lambda_1, \ldots, \lambda_n$$

$$\text{(i+1)th raw: } -(\delta_1, \delta_{1+i}), \ldots, -(\delta_n, \delta_{n+1});\ 1 \le i \le [n/2]$$

$$\text{j th column: } (\lambda_j, (\delta_j, \delta_{j+1}), \ldots, (\delta_j, \delta_{j+[n/2]}))^t;\ 1 \le j \le n(mod\, n)$$

Let us illustrate this by giving a specific example:

[2]The presentation of this section is based on [3].

**Example 1** Let $\delta_1$, $\delta_2$, $\delta_3$, $\delta_4$, $\delta_5$ be elements with positive squares, each of them orthogonal to exactly one side of a convex polytope, of five sides, in hyperbolic space. So, $1 \le i \le [5/2]$ and:

$$G(A) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \delta_1\,\delta_2 & \delta_2\,\delta_3 & \delta_3\,\delta_4 & \delta_4\,\delta_5 & \delta_5\,\delta_1 \\ \delta_1\,\delta_3 & \delta_2\,\delta_4 & \delta_3\,\delta_5 & \delta_4\,\delta_1 & \delta_5\,\delta_2 \end{pmatrix}$$

Our problem is to find all matrices $G(A)$, having the lattice Weyl vector $\rho$, $\rho \in M \otimes \mathbb{Q}$, such that $(\rho, \alpha) = -(\alpha, \alpha)/2 \Leftrightarrow (\rho, \delta_i) = -\lambda_i$, $i = 1, \ldots, n$. The answer has been presented by Gritsenko and Nikulin in the following theorem (1.2.1. from [3]):

**Theorem 1** *All geometric realisations $G(A)$, of hyperbolic generalised Cartan matrices $A$ of rank 3, of elliptic type, with the lattice Weyl vector, which are twisted to symmetric generalised Cartan matrices, all twisting coefficients $\lambda_i$ satisfy:*

$$\lambda_i \le 12$$

*are given in Table 1 (from [3]).*

**Remark 5** Table 1 (from [3]) gives 60 matrices. 7 of them are of the compact case (they represent a convex polytope of finite volume in hyperbolic space), the 4 of which are untwisted. The rest 53 matrices are of the non-compact case.

All these matrices can be found in pages 166-168 , from [3].

The main aim of this section is to discuss about the conjecture that follows the above theorem, which we give below:

**Conjecture 1** Table 1 (from [3]) gives the complete list of hyperbolic generalised Cartan matrices $A$ of rank 3, of elliptic type, with the lattice Weyl vector, which are twisted to symmetric generalised Cartan matrices.

In other words, one can drop the inequality $\lambda_i \le 12$ from theorem 1.2.1., from [3].

Gritsenko and Nikulin give, in the same paper, the following arguments for supporting the conjecture:

(i) The number of all hyperbolic generalised Cartan matrices of elliptic type, with the lattice Weyl vector, is finite, for rank greater than or equal to 3.

So, there exists an absolute constant $m$, such that $\lambda_i \le m$.

(ii) Calculations were done for all $\lambda_i \leq 12$, but result has only matrices with all $\lambda_i \leq 6$.

So, there do not exist new solutions between 6 and 12.

We will now propose a way of attacking the mentioned conjecture, in four steps, giving the complete solution for the first proposed step.

FIRST STEP: We consider a triangle in hyperbolic space, with sides $a$, $b$ and $c$. The angle between $a$ and $b$ is $\pi/2$, between $a$ and $c$ is $\pi/3$ and between $c$ and $b$ is 0 radians, that is, the vertex which is created from the intersection of $c$ and $b$ is at infinity of our space. This triangle will be the fundamental chamber for reflection in $\Lambda^+(M)$, and the reflections will cover all space, tending to infinity.

In the proof of theorem 1.2.1, the authors gave the following relations, which we will use:

$$0 \leq (\delta_1, \delta_2) \leq 2, \; 0 \leq (\delta_1, \delta_3) < 14, \; 0 \leq (\delta_2, \delta_3) \leq 2 \quad (*),$$

for $\delta_1$, $\delta_2$, $\delta_3 \in \tilde{P}(\mathcal{M})$ being orthogonal vectors to three consecutive sides, of a polygon $A_1 \dots A_n$ in $\Lambda^+(M)$, namely $A_1A_2$, $A_2A_3$, $A_3A_4$.

Our suggestion is to find all these $\delta_1$, $\delta_2$, $\delta_3$ satisying $(*)$, for the group generated by reflections in $\Lambda^+(M)$, with fundamental chamber the triangle with sides $a$, $b$, $c$.

We fix $\delta_2 = a$. Then, we have the following possibilities for $\delta_1$ :

$$\delta_1 = c$$

$$s_c(b) = b - 2((b,c)/c^2)c = b + 2c$$

$$s_{b+2c}(a) = a - 2((a, b+2c)/(b+2c)^2)(b+2c) = a + 2b + 4c$$

etc.

In other words:

$$\delta_1 = na + (n+1)b + 2(n+1)c$$

Now, the posibilities for $\delta_3$ are:

$$s_b(c) = c - 2((c,b)/b^2)b \Rightarrow \delta_3 = 2b + c$$

$$s_{2b+c}(-b) = -b - 2((b, 2b+c)/(2b+c)^2)(2b+c) \Rightarrow \delta_3 = 3b + 2c$$

Also:

$$s_{3b+2c}(a) = a - 2((a, 3b+2c)/(3b+2c)^2)(3b+2c) = a + 6b + 4c$$

$$s_{a+6b+4c}(-3b-2c) = (-3b-2c)-2((-3b-2c, a+6b+4c)/(a+6b+4c)^2)(a+6b+4c) =$$
$$2a+9b+6c$$

etc.

So,

$$\delta_3 = (n+1)a + (3n+6)b + (2n+4)c$$

**Remark 6** We fixed $\delta_2 = a$, and we looked for possible $\delta_1$ and $\delta_3$, such that the angle between $\delta_2$ and $\delta_1$ is accute; the same for the angle between $\delta_2$ and $\delta_3$.

Below, we give the list of all possible $\delta_1$ and $\delta_3$, for $\delta_2 = a$, such that $(\delta_1, \delta_3) < 14$ (in the right hand side we give $\delta_3$ and in the left $\delta_1$) :

$$c,\ b$$

$$b+2c,\ b$$
$$a+2b+4c,\ b$$
$$2a+3b+6c,\ b$$
$$3a+4b+8c,\ b$$
$$4a+5b+10c,\ b$$
$$5a+6b+12c,\ b$$

$$c,\ 2b+c$$
$$c,\ 3b+2c$$

$$c,\ a+6b+4c$$
$$c,\ 2a+9b+6c$$
$$c,\ 3a+12b+8c$$

$$b+2c,\ 2b+c$$
$$a+2b+4c,\ 2b+c$$
$$2a+3b+6c,\ 2b+c$$
$$3a+4b+8c,\ 2b+c$$

$$b+2c,\ 3b+2c$$

$$a+2b+4c,\ 3b+2c$$

$$2a+3b+6c,\ 3b+2c$$

$$b+2c,\ a+6b+4c$$

$$b+2c,\ 2a+9b+6c$$

We now fix $\delta_2 = b$, and give the list of all possible $\delta_1$ and $\delta_3$, for $\delta_2 = b$, such that $(\delta_1, \delta_3) < 14$ (in the right hand side we give $\delta_3$ and in the left $\delta_1$) :

$$c,\ a$$

$$b+2c,\ a$$

$$2b+3c,\ a$$

$$3b+4c,\ a$$

$$4b+5c,\ a$$

$$5b+6c,\ a$$

$$6b+7c,\ a$$

$$7b+8c,\ a$$

$$8b+9c,\ a$$

$$9b+10c,\ a$$

$$10b+11c,\ a$$

$$11b+12c,\ a$$

$$12b+13c,\ a$$

$$c,\ a+c$$

$$c,\ 2a+b+2c$$

$$c,\ 3a+2b+3c$$

$$c,\ 4a+3b+4c$$

$$c,\ 5a+4b+5c$$
$$c,\ 6a+5b+6c$$
$$c,\ 7a+6b+7c$$
$$c,\ 8a+7b+8c$$
$$c,\ 9a+8b+9c$$
$$c,\ 10a+9b+10c$$
$$c,\ 11a+10b+11c$$
$$c,\ 12a+11b+12c$$
$$c,\ 13a+12b+13c$$
$$c,\ 14a+13b+14c$$
$$c,\ 15a+14b+15c$$

$$b+2c,\ a+c$$

$$b+2c,\ 2a+b+2c$$
$$b+2c,\ 3a+2b+3c$$
$$b+2c,\ 4a+3b+4c$$
$$b+2c,\ 5a+4b+5c$$
$$b+2c,\ 6a+5b+6c$$
$$b+2c,\ 7a+6b+7c$$

$$2b+3c,\ a+c$$

$$2b+3c,\ 2a+b+2c$$
$$2b+3c,\ 3a+2b+3c$$
$$2b+3c,\ 4a+3b+4c$$
$$2b+3c,\ 5a+4b+5c$$

$$3b+4c,\ a+c$$

$$3b+4c,\ 2a+b+2c$$
$$3b+4c,\ 3a+2b+3c$$

$$4b+5c,\ \ a+c$$

$$4b+5c,\ \ 2a+b+2c$$

$$4b+5c,\ \ 3a+2b+3c$$

$$5b+6c,\ \ a+c$$

$$5b+6c,\ \ 2a+b+2c$$

$$6b+7c,\ \ a+c$$

$$6b+7c,\ \ 2a+b+2c$$

$$7b+8c,\ \ a+c$$

$$7b+8c,\ \ 2a+b+2c$$

$$8b+9c,\ \ a+c$$

$$9b+10c,\ \ a+c$$

$$10b+11c,\ \ a+c$$

$$11b+12c,\ \ a+c$$

$$12b+13c,\ \ a+c$$

$$13b+14c,\ \ a+c$$

$$14b+15c,\ \ a+c$$

Last, we fix $\delta_2 = c$, and give the list of all possible $\delta_1$ and $\delta_3$, for $\delta_2 = c$, such that $(\delta_1, \delta_3) < 14$ (in the right hand side we give $\delta_3$ and in the left $\delta_1$) :

$$b,\ a$$

$$2b+c,\ a$$

$$3b+2c,\ a$$

$$4b+3c,\ a$$

$$5b+4c,\ a$$

$$6b+5c,\ a$$

$$7b+6c,\ a$$

$$8b+7c,\ a$$

$$9b+8c,\ a$$

$$10b+9c,\ a$$

$$11b+10c,\ a$$

$$12b+11c,\ a$$

$$13b+12c,\ a$$

$$14b+13c,\ a$$

$$b,\ 2a+b+2c$$

$$b,\ 8a+6b+9c$$

$$b,\ 12a+9b+14c$$

$$b,\ 16a+12b+19c$$

$$2b+c,\ 2a+b+2c$$

$$2b+c,\ 3a+2b+3c$$

$$2b+c,\ 4a+3b+4c$$

$$2b+c,\ 5a+4b+5c$$

$$2b+c,\ 6a+5b+6c$$

$$2b+c,\ 7a+6b+7c$$

$$2b+c,\ 8a+7b+8c$$

$$2b+c,\ 9a+8b+9c$$

$$2b+c,\ 10a+9b+10c$$

$$2b+c,\ 11a+10b+11c$$

$$3b+2c,\ 2a+b+2c$$

$$3b+2c,\ 3a+2b+3c$$

$$3b+2c,\ 4a+3b+4c$$

$$3b+2c,\ 5a+4b+5c$$

$$4b+3c,\ 2a+b+2c$$

$$4b+3c,\ 3a+2b+3c$$

$$5b+4c,\ 2a+b+2c$$

$$5b+4c,\ 3a+2b+3c$$

$$6b+5c,\ 2a+b+2c$$

SECOND STEP: So, we have found 115 triples of elements $\delta_1$, $\delta_2$, $\delta_3$, as we described above, and we now need to find, for each case separately, the correspoding twisting coefficients, $\lambda_1$, $\lambda_2$, $\lambda_3$. So, we introduce three new elements, $\tilde{\delta_1} = \lambda_1\,\delta_1$, $\tilde{\delta_2} = \lambda_2\,\delta_2$, $\tilde{\delta_3} = \lambda_3, \delta_3$

As we mentioned in the first section, any $\tilde{\delta_i}$ and $\tilde{\delta_j}$ should satisfy the relations:

$$\tilde{\delta_i}^2 |\, 2(\tilde{\delta_i}, \tilde{\delta_j}) \Rightarrow 2\lambda_i |\, 2\lambda_j(\delta_i, \delta_j)\ (**)$$

$$\tilde{\delta_j}^2 |\, 2(\tilde{\delta_i}, \tilde{\delta_j}) \Rightarrow 2\lambda_j |\, 2\lambda_i(\delta_i, \delta_j)$$

Now, let us introduce some more notation; so, by $\nu_p(\lambda)$, we denote the power of the prime number $p$, in the prime factorisation of the natural number $\lambda$. For example, $\nu_3(21) = 1$ and $\nu_2(21) = 0$.

Applying this notation to our case, we get that for our $\tilde{\delta_1}$, $\tilde{\delta_2}$, $\tilde{\delta_3}$, the following conditions should be satisfied:

$$|\nu_p(\lambda_1) - \nu_p(\lambda_2)| \leq \nu_p(g_{12})$$

$$|\nu_p(\lambda_1) - \nu_p(\lambda_3)| \leq \nu_p(g_{13})$$

$$|\nu_p(\lambda_2) - \nu_p(\lambda_3)| \leq \nu_p(g_{23})$$

where $g_{12}$, $g_{13}$, $g_{23}$ are elements of the Gram matrix of $\delta_1$, $\delta_2$, $\delta_3$, with diagonal equal to 2 and (by definition) $g_{ij} = g_{ji}$.

So, the information that we have is quite enough, for calculating $\lambda_1$, $\lambda_2$, $\lambda_3$, for each of the 115 triples that we calculated in (i), simply by working on the Gram matrix for each case.

Another way to do this calculations is via programming. Here we introduce a programme in MAPLE (with some explanation following the programme):

```
with(numtheory);
[GIgcd, bigomega, cfrac, cfracpol, cyclotomic, divisors, factorEQ, factorset,
fermat, imagunit, index, integral_basis, invcfrac, invphi, issqrfree,
jacobi, kronecker, lambda, legendre, mcombine, mersenne, migcdex, minkowski,
mipolys, mlog, mobius, mroot, msqrt, nearestp, nthconver, nthdenom,
nthnumer, nthpow, order, pdexpand, phi, pi, pprimroot, primroot, quadres,
rootsunity, safeprime, sigma, sq2factor, sum2sqr, tau, thue]
x_y:=2: x_z:=4: y_x:=2: y_z:=2: z_x:=4: z_y:=2:
count:=0:
for x in divisors(lcm(x_y*x_z))do
for y in divisors(lcm(y_x*y_z)) do
for z in divisors(lcm(z_x,z_y)) do
if modp(x_y*y,x)=0 and
modp(x_z*z,x)=0 and
modp(y_x*x,y)=0 and
modp(y_z*z,y)=0 and
modp(z_x*x,z)=0 and
modp(z_y*y,z)=0 and igcd(x,y,z)=1
then print(x,y,z);
count:=count+1;
fi;
od;
od;
od;
printf ("Number of solutions
1, 1, 1
1, 1, 2
1, 2, 1
1, 2, 2
1, 2, 4
```

2, 1, 1
2, 1, 2
2, 2, 1
4, 2, 1
Number of solutions 9

This programme works as follows; for each $\delta_1$, $\delta_2$, $\delta_3$ we apply $(**)$ and get six relations of the type:

$$x|x\_y * y$$

$$x|x\_z * z$$

$$y|y\_x * x$$

$$y|y\_z * z$$

$$z|z\_x * x$$

$$z|z\_y * y$$

where $\gcd(x, y, z) = 1$. It is enough to examine the values of $x$, which belong to the set $S_x$, of the divisors of the $\text{lcm}(x\_y * x\_z)$. In other words, $\text{lcm}(x, y, z) = x$. The same for $y$ and $z$, we have the sets $S_y$ and $S_z$ respectively, and check for which triples $(x, y, z)$ (with $x$, $y$, $z$ be longing to $S_x$, $S_y$, $S_z$ respactively) the conditions hold.

THIRD STEP: The third step is to find for each triple $\delta_1$, $\delta_2$, $\delta_3$ the lattice Weyl vector. And we already mentioned, in the first section, that the following relation should be satisfied:

$$(\rho, \delta_i) = -\lambda_i,\ 1 \leq i \leq 3$$

Furthermore, it should be always true that $(\rho, \rho) < 0$.

We are optimistic that we will have a computer programme which can do this calculation, soon.

FOURTH STEP: In this step we will try to find more elements $\delta_i$, which will be orthogonal to sides of polytopes in $\Lambda^+(M)$; polytopes to which $\delta_1$, $\delta_2$, $\delta_3$ were orthogonal (each one perpendicular to a side of the polytope, respectively).

So, for each case (from steps (i)-(iii)) we should examine if one more side exists, where a new element $\delta_4$ is orthogonal to this side. It will be vital to prove the existence of the lattice Weyl vector.

What we already know is that the determinant of the Gram matrix of $\delta_1$, $\delta_2$, $\delta_3$, $\delta_4$ should be equal to zero, because $\delta_1$, $\delta_2$, $\delta_3$ should come from a $3-$D lattice, and so the four elements $\delta_1$, $\delta_2$, $\delta_3$, $\delta_4$ should be linearly independent.

Also, $(**)$ should take the form:

$$(\lambda_4\,\delta_4)^2 |\, 2(\lambda_4\,\delta_4, \lambda_i\,\delta_i),$$

where $\delta_4 = x_1\delta_1 + x_2\delta_2 + x_3\delta_3$, $x_i \in \mathbb{Q}$.

For the cases where $\delta_4$ does not exist, we stop there. If it exist, then we repeat the same thinking for a new element, $\delta_5$ and so on.

We will have proved the conjecture to theorem 1.1.2, from [1], when we find all convex polytopes in the hyperbolic space, which correspond to the matrices from Table 1 (from [3]).

## 0.3 Some Basic Definitions on Lie Algebras

Before talking about Kac-Moody algebras, we find it useful to recall some basic notions on Lie-algebras. Let us start by defining a *Lie algebra*, $\mathfrak{g}$ :

**Definition 14** *Lie algebra* is the algebra $\mathfrak{g}$, over a field $F$; $(x, y) \to [x, y]$, $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that:

(i) $[x, y] = -[y, x]$ (anticommutativity),

(ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity), for all $x$, $y \in \mathfrak{g}$.

Such an algebra, $\mathfrak{g}$, is said to be *associative*, if and only if $[x, y] = xy - yx$ and *commutative*, if $[x, y] = [y, x]$.

A subspace $\mathfrak{h}$, of a Lie algebra $\mathfrak{g}$, is a *Lie subalgebra* of $\mathfrak{g}$, if and only if:

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \Leftrightarrow [x, y] \in \mathfrak{h},$$

where $x, y \in \mathfrak{h}$.

Furthermore, we define the *derived algebra* $\mathfrak{g}'$, *of* $\mathfrak{g}$, as follows:

$$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$$

and a *derivation* $\delta(xy)$, of $\mathfrak{g}$ :

$$\delta(xy) = \delta x\, y + x\, \delta y,$$

where $\operatorname{Der}\mathfrak{g}$ is the set of all derivations, and a Lie subalgebra, of $\mathfrak{g}$.

We now define the *adjoint representation of* $\mathfrak{g}$ as follows:

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ x \to \operatorname{ad} x,$$

[2]The presentation of this section is based on [4].

$$\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g} \text{ (homomorphism)}, \ (\mathrm{ad}\, x)y = [x, y].$$

A Lie subalgebra $\mathfrak{h}$ is called *ideal* of $\mathfrak{g}$, if and only if:

$$[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$$

and $\mathfrak{g}$ is called *simple*, if its only ideals are the zero subalgebra and itself, the *trivial ideals*.

Last, but not least, we define the *centre* $Z(\mathfrak{g})$, of $\mathfrak{g}$ :

$$Z(\mathfrak{g}) = \{g \in \mathfrak{g} : [x, g] = 0, \ x \in \mathfrak{g}\}$$

## 0.4 Symmetrizable Kac-Moody Algebras

We now have the machinery to start constructing *symmetrizable Kac-Moody algebras.*

We consider the *root lattice* $Q = \bigoplus_{i=1}^{n} \mathbb{Z}\, \alpha_i$ and the *root semigroup* (which is an integral cone) $Q_+ = \sum_{i=1}^{n} \mathbb{Z}^+\, \alpha_i \subseteq Q$, where $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{h}$, where $\mathfrak{h}$ is a complex vector space.

Moreover, we consider the *coroot lattice* $Q^* = \bigoplus_{i=1}^{n} \mathbb{Z}\, \alpha_i^*$ and the *coroot semigroup* $Q_+^* = \sum_{i=1}^{n} \mathbb{Z}^+\, \alpha_i^* \subseteq Q^*$, where $\{\alpha_1^*, \ldots, \alpha_n^*\} \subseteq \mathfrak{h}^*$.

In addition, we will work on a natural pairing:

$$<,>: Q^* \times Q \to \mathbb{Z}, \ \ < \alpha_i^*, \alpha_j >= a_{ij}$$

and on an integral quadratic form (which is said to be canonical):

$$(,) : Q \times Q \to \mathbb{Z}, \ (\alpha_i, \alpha_j) = b_{ij} = a_{ij}/\epsilon_i,$$

where $a_{ij}$, $b_{ij}$, $\epsilon_i$ were defined in the first section, and:

$$a_{ij} =< \alpha_i^*, \alpha_j >= 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$$

**Definition 15** *Kac-Moody algebra*, $\mathfrak{g}'(A)$, is the complex Lie algebra defined by $3n$ generators $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, $\alpha_1^*, \ldots, \alpha_n^*$ and the defining relations:

(i) $[\alpha_i^*, \alpha_j^*] = 0$,

(ii) $[e_i, f_j] = \delta_{ij}\, \alpha_i^*$,

(iii) $[\alpha_i^*, e_j] = a_{ij}\, e_j$,

(iv) $[\alpha_i^*, f_j] = -a_{ij}\, f_j$ and

[2]The presentation of this section is based on [5] and on [6].

(v) $(\mathrm{ad}\, e_i)^{1-a_{ij}}\, e_j = 0$,
$(\mathrm{ad}\, f_i)^{1-a_{ij}}\, f_j = 0$, where $\delta_{ij}$ is the Kronecker symbol, $i \neq j$ and $1 \leq i \leq n$.

We denote the *maximal commutative subalgebra* of $\mathfrak{g}'(A)$ by $\mathfrak{h}'$, where $\mathfrak{h}' = Q^* \otimes \mathbb{C}$, and it can be easily shown that $Z(\mathfrak{g}'(A)) = \{x \in \mathfrak{h}' : < x, Q >= 0\}$ and $\mathfrak{g}'(A)/Z(\mathfrak{g}'(A))$ is simple.

The final step is to construct the *root space decomposition* of $\mathfrak{g}'(A)$, and we do this as follows:

$$\mathfrak{g}'(A) = \bigoplus_{\alpha_i \in Q_+} \mathfrak{g}_{-\alpha_i} \oplus \mathfrak{h}' \oplus \bigoplus_{\alpha_i \in Q_+} \mathfrak{g}_{\alpha_i},$$

where $\mathfrak{g}_{\alpha_i} = \{x \in \mathfrak{g}'(A) : [h, x] =< h, \alpha_i > x,\ h \in \mathfrak{h}'\}$, where the dimension of $\mathfrak{g}_{\alpha_i}$ is finite, and it is called *the multiplicity of* $\alpha_i$, $\alpha_i \in \pm Q_+$. This $\alpha_i$ is called *root*, if and only if its multiplicity is greater than zero. We denote by $\Delta$ a *set of roots*, $\Delta = \Delta_+ \cup -\Delta_+$, (disjoint union) where $\Delta_+ = \Delta \cap Q_+$.

The description of roots and their multiplicities is an important problem for Kac-Moody algebras. We will now give an account of Kac's description.

Let $\alpha_1, \ldots, \alpha_n$ be roots. We call these roots *simple roots*. Let also $s_{\alpha_i} \in GL(Q)$ be fundamental reflections and $W$ the group generated by all $s_{\alpha_i}$, and preserving the canonical integral quadratic form $(,)$.

Then, $\Delta$ is invariant with respect to $W$ and is divided into $\Delta^{\mathrm{re}}$ and $\Delta^{\mathrm{im}}$, in other words it is divided into real and imaginary roots, where:

$$\Delta^{\mathrm{re}} := W(\alpha_1) \cup \ldots \cup W(\alpha_n),$$

$$\Delta^{\mathrm{im}} := W(K),\ K = \{\alpha_i \in Q_+ - \{0\} : (\alpha, \alpha_i) \leq 0\, \forall\, \alpha_i,\ \mathrm{supp}\ \alpha \text{ connected}\}.$$

For $\alpha = \sum_{i=1}^n k_i \alpha_i \in Q_+$, $\mathrm{supp}\, \alpha \subseteq \{\alpha_1, \ldots, \alpha_n\}$ is the set of all $\alpha_i$, such that $k_i > 0$, and is said to be *connected*, if there does not exist a decomposition $\mathrm{supp}\, \alpha = A_1 \cup A_2$, such that $(A_1, A_2) = 0$, where $A_1$, $A_2$ are non-empty sets.

**Remark 7** Obviously, for $\alpha \in \Delta^{\mathrm{re}} \Rightarrow (\alpha, \alpha) > 0$ and for $\alpha \in \Delta^{\mathrm{im}} \Rightarrow (\alpha, \alpha) \leq 0$ and also $n\, \alpha \in \Delta^{\mathrm{im}}$, $n \in \mathbb{N}$.

# Bibliography


[1] Viacheslav V. Nikulin *Reflection Groups in Hyperbolic Spaces and the Denominator Formula for Lorentzian Kac-Moody Algebras,* Izv. Math., 1996, 60 (2), 305-334.

[2] E.B. Vinberg *Hyperbolic Reflection Groups,* Russian Math. Surveys 40:1 (1985), 31-75.

[3] Valeri A. Gritsenko, Viacheslav V. Nikulin *Automorphic Forms and Lorentzian Kac-Moody Algebras, Part 1,* International Journal of Mathematics, Volume 9, No. 2 (1998).

[4] James E. Humphreys *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag (1980).

[5] Viacheslav V. Nikulin *A Lecture on Kac-Moody Lie Algebras of Arithmetic Type ,* Preprint alg-geom/9412003.

[6] Victor G. Kac *Infinite Dimensional Lie Algebras,* Cambridge University Press, third edition (1990).