A SHORT INTRODUCTION TO TORIC VARIETIES

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Abstract. Toric varieties form an important class of algebraic varieties whose particular strength lies in methods of construction via combinatorial data. Understanding this construction has led to the development of a rich dictionary allowing combinatorial statements to be translated into algebraic statements, and vice versa.

In this short summary, the basic details of the combinatorial approach to constructing toric varieties are given. The constructions are motivated by specific examples from which the more general methods can be deduced.

0. Introduction

Toric varieties form an important class of algebraic varieties. The structure of a toric variety is intimately connected with a corresponding combinatorial description, allowing one to easily illustrate such concepts as linear systems, invertible sheaves, cohomology, and resolution of singularities.

The goal of this short note is to describe the main constructions of toric varieties, highlighting the rich combinatorial structure which makes their study so rewarding. As an introduction, our approach will differ from the standard reference [Ful93]; less emphasis will be placed on proofs of combinatorial statements than might ordinarily be the case.

A substantial body of introductory literature exists on the subject of toric varieties. In addition to the concise and instructive work [Ful93] already mentioned, [Ewa96] and [Oda78] are invaluable. Of equal merit are [BB02] and [Dan78]. For a survey of the current state of research in this field, consult [Cox02].

These notes are intended to supplement a talk given by the author to the "Calf" (junior **C**ambridge **O**xford **W**arwick) seminar in November 2004.

1. The fan associated with $\mathbb{P}^2$

Before a definition of what it means for an algebraic variety to be a toric variety is given, let us consider the (complex) projective plane $\mathbb{P}^2$. This can be realised as a collection of three isomorphic copies of $\mathbb{C}^2$ glued together in the appropriate fashion.

More specifically if $(z_0 : z_1 : z_2)$ are taken to be the homogeneous coordinates of $\mathbb{P}^2$, the coordinate charts:

$$U_i := \{(z_0/z_i : z_1/z_i : z_2/z_i) \mid z_i \neq 0\} \qquad i = 0, 1, 2$$

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are each isomorphic to $\mathbb{C}^2$. These charts glue together via maps of the form:

$$\begin{aligned}\phi_{0,1} : U_0 \setminus (z_1/z_0 = 0) &\to U_1 \setminus (z_0/z_1 = 0)\\ (1 : x : y) &\mapsto (1/x : 1 : y/x).\end{aligned}$$

Associated to each chart U_i is the ring of regular functions, $\mathbb{C}[U_i]$;

$$\mathbb{C}[U_0] = \mathbb{C}[z_1/z_0, z_2/z_0], \qquad \mathbb{C}[U_1] = \mathbb{C}[z_0/z_1, z_2/z_1], \qquad \mathbb{C}[U_2] = \mathbb{C}[z_0/z_2, z_1/z_2].$$

By setting $X := z_1/z_0$ and $Y := z_2/z_0$ we obtain:

$$\mathbb{C}[U_0] = \mathbb{C}[X, Y], \qquad \mathbb{C}[U_1] = \mathbb{C}[X^{-1}, X^{-1}Y], \qquad \mathbb{C}[U_2] = \mathbb{C}[Y^{-1}, XY^{-1}].$$

Each $\mathbb{C}[U_i]$ is contained within the coordinate ring $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$. This is the coordinate ring associated with the *algebraic torus* $(\mathbb{C}^*)^2 \cong \operatorname{Spec}(\mathbb{C}[X^{\pm 1}, Y^{\pm 1}])$, from which toric varieties derive their name.

Let $M \cong \mathbb{Z}^2$ be the lattice of Laurent monomials in X and Y. Thus points in M correspond to monomials $X^m Y^n$ for some $(m, n) \in \mathbb{Z}^2$. Let $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^2$. We regard monomials in the coordinate rings $\mathbb{C}[U_i]$ as lattice points in $M_\mathbb{R}$. Thus $\mathbb{C}[U_0]$ corresponds to the cone[1] with generators $\{X, Y\}$ (i.e. the nonnegative quadrant of $M_\mathbb{R}$), $\mathbb{C}[U_1]$ corresponds to the cone with generators $\{X^{-1}, X^{-1}Y\}$, and $\mathbb{C}[U_2]$ to that with generators $\{Y^{-1}, XY^{-1}\}$. Thus to each coordinate ring $\mathbb{C}[U_i]$ we have associated a cone in $M_\mathbb{R}$, denoted $\sigma_i^\vee$.

To each cone $\sigma_i^\vee$ we associate its *dual cone*[2] σ_i. These dual cones will lie in the vector space $N_\mathbb{R} \cong \mathbb{R}^2$ obtained from the lattice $N := \operatorname{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^2$. To be precise:

$$\sigma_i := \left\{v \in N_\mathbb{R} \mid \langle u, v\rangle \geq 0 \text{ for all } u \in \sigma_i^\vee\right\}.$$

This gives cones σ_0, σ_1 and σ_2 in $N_\mathbb{R}$ with generators $\{(1,0),(0,1)\}, \{(0,1),(-1,-1)\}$ and $\{(1,0),(-1,-1)\}$ respectively.

Let Δ denote the collection consisting of the two-dimensional cones σ_i $(i = 0, 1, 2)$, the three one-dimensional cones generated by $(1,0),(0,1)$ and $(-1,-1)$, and the zero-dimensional cone formed by the origin. The resulting cell complex Δ is called the *fan associated with* $\mathbb{P}^2$.

The structure of the fan Δ reflects the structure of $\mathbb{P}^2$. We see three two-dimensional cones, which correspond to the two-dimensional subvarieties U_0, U_1 and U_2. Each two-dimensional cone σ_i is glued to σ_j along a one-dimensional cone; these correspond to the subvarieties $U_i \cap U_j$, for $i \neq j$. Finally, all the cones are glued together in the zero-dimensional cone given by the origin. This zero-dimensional cone corresponds to the algebraic torus, and "is" the subvariety $U_0 \cap U_1 \cap U_2$.

[1]A *cone* (or more precisely a *finitely generated rational polyhedral cone*) in $M_\mathbb{R}$ is a set of the form $\left\{\sum_{i=1}^k \lambda_i u_i \in M_\mathbb{R} \mid \lambda_i \geq 0\right\}$ for some finite collection of elements $u_1, \ldots, u_k$ in M.

[2]The dual to a cone σ is usually denoted by $\sigma^\vee$. Since it can easily be shown (eg. [Ful93, pg. 9]) that $(\sigma^\vee)^\vee = \sigma$, the expression above conforms to this notation.

2. Cones and Fans

The procedure in Section 1 is reversible, and the methods involved generalise to any fan. Indeed, the driving force behind the study of toric varieties is the fact that fans and toric varieties are in one-to-one correspondence. Before the method of constructing a toric variety from a fan is described, a definition of what it means for a variety to be a toric variety is long overdue.

Definition 2.1. A *toric variety* of dimension n over an algebraically closed field $k = \overline{k}$ is a normal variety X that contains a torus $T \cong (k^*)^n$ as a dense open subset, together with an action $T \times X \to X$ of T on X that extends the natural action of T on itself.

Let $M \cong \mathbb{Z}^n$ be a lattice, and $N = \mathrm{Hom}\,(M, \mathbb{Z}) \cong \mathbb{Z}^n$ be its dual lattice. In $N_{\mathbb{R}}$ we define a fan Δ.

Definition 2.2. A *fan* Δ is defined to be a finite collection of cones in $N_{\mathbb{R}}$ such that:

(i) If $\sigma \in \Delta$, then $\sigma \cap (-\sigma) = \{0\}$. Such a cone is said to be *strongly convex*[3].
(ii) If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.
(iii) If $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma' \in \Delta$.

Note that the fan derived in Section 1 satisfies this definition. It is possible run the construction in reverse and recover the unique toric variety associated to each fan.

For each cone $\sigma \in \Delta$ there exists a dual cone $\sigma^\vee \subset M_{\mathbb{R}}$. We define the semigroup $S_\sigma := \sigma^\vee \cap M$. This semigroup is finitely generated, by *Gordon's Lemma* [Ful93, pg. 12].

We now define the corresponding affine ring $A_\sigma := \mathbb{C}[S_\sigma]$. We denote by χ^u the element in the $\mathbb{C}$-algebra corresponding to the semigroup element $u \in S_\sigma$. We require that $\chi^u \chi^{u'} := \chi^{u+u'}$. The elements of $\mathbb{C}[S_\sigma]$ are thus given by finite sums $\sum c_i \chi^{u_i}$, where $c_i \in \mathbb{C}, u_i \in S_\sigma$.

Finally, the affine variety U_σ corresponding to a cone σ is given by:

$$U_\sigma := \mathrm{Spec}\,(\mathbb{C}[S_\sigma])\,.$$

These affine varieties can be glued together via the following Lemma:

Lemma 2.3. *If τ is a face of σ then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

Proof. See [Ful93, pg. 18]. □

Since any two cones $\sigma, \sigma' \in \Delta$ share a common face, there are injections $\phi : U_{\sigma\cap\sigma'} \to U_\sigma$ and $\varphi : U_{\sigma\cap\sigma'} \to U_{\sigma'}$. The identification is then given by:

$$\begin{aligned} f : \phi(U_{\sigma\cap\sigma'}) &\to \varphi(U_{\sigma\cap\sigma'}) \\ x &\mapsto \varphi(\phi^{-1}(x)), \end{aligned}$$

[3] A cone is strongly convex if it contains no one-dimensional vector space.

with inverse $y \mapsto \phi(\varphi^{-1}(y))$. Patching for all $\sigma \in \Delta$ gives the toric variety denoted by X_Δ.[4]

Note that since $\{0\} \subset N_\mathbb{R}$ is a face of every cone in Δ, so $U_{\{0\}}$ can be regarded as sitting inside each affine variety U_σ. Now $S_{\{0\}} = M$, where we regard the lattice M as a semigroup with $2n$ generators. In particular if N is generated (as a lattice) by $e_1, \ldots, e_n$, then M is generated (as a semigroup) by $\pm e_1^*, \ldots, \pm e_n^*$. Setting $X_i := \chi^{e_i^*}$ and $X_i^{-1} := \chi^{-e_i^*}$ we see that:

$$U_{\{0\}} = \mathrm{Spec}\left(\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]\right) \cong (\mathbb{C}^*)^n.$$

Thus the torus is a principal open subset of all the U_σ. For more details consult any of the introductory texts listed in Section 0. This is why X_Δ is called a toric variety.

3. Examples

Example 3.1. Consider the two-dimensional cone $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^2$ generated by e_2 and $2e_1 - e_2$, where e_1, e_2 is a basis for N. The dual cone is generated by e_1^* and $e_1^* + 2e_2^*$, however the corresponding semigroup $S_\sigma := \sigma^\vee \cap M$ has three generators, $\chi^{e_1^*}, \chi^{e_1^*+2e_2^*}$, and $\chi^{e_1^*+e_2^*}$. Writing $X := \chi^{e_1^*}$ and $Y := \chi^{e_2^*}$, we see that $\mathbb{C}[U_\sigma] = \mathbb{C}[X, XY, XY^2]$.

Under the ring homomorphism $\mathbb{C}[u, v, w] \to \mathbb{C}[S_\sigma]$ defined by:

$$u \mapsto X, v \mapsto XY, w \mapsto XY^2,$$

we observe that $uw - v^2 = 0$, since $e_1^* + (e_1^* + 2e_2^*) = 2(e_1^* + e_2^*)$. Hence we have that $A_\sigma = \mathbb{C}[u, v, w]/\left\langle uw - v^2 \right\rangle$, and so:

$$U_\sigma = \left\{(u, v, w) \in \mathbb{C}^3 \mid uw = v^2\right\}.$$

This is the affine cone over a conic, also known as the du Val singularity of type A_1.

Example 3.2. Consider the three-dimensional cone $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^3$ generated by e_1, e_2, e_3, and $e_1 + e_3 - e_2$ (where e_1, e_2, e_3 form a basis of N). This has dual cone $\sigma^\vee$ generated by:

$$u_1 := e_1^*, u_2 := e_3^*, u_3 := e_1^* + e_2^*, \text{ and } u_4 := e_2^* + e_3^*.$$

Setting $X_i := \chi^{e_i^*}$ we see that $A_\sigma = \mathbb{C}[X_1, X_3, X_1X_2, X_2X_3]$.

Under the ring homomorphism $\mathbb{C}[x, y, z, w] \to \mathbb{C}[S_\sigma]$ defined by:

$$w \mapsto X_1, x \mapsto X_3, y \mapsto X_1X_2, \text{ and } z \mapsto X_2X_3,$$

one sees that $wz - xy \mapsto 0$ (since $u_1 + u_4 = u_2 + u_3$). Hence $\mathbb{C}[w, x, y, z]/\langle wz - xy\rangle \cong \mathbb{C}[S_\sigma]$, and so U_σ is the hypersurface defined by $wz = xy$ in $\mathbb{C}^4$.

[4]The patching can also be constructed at the level of $\mathbb{C}$-algebras. For any cones $\sigma, \sigma' \in \Delta$ there exists $u \in (-\sigma')^\vee \cap \sigma^\vee$ such that $\sigma \cap u^\perp = \sigma' \cap \sigma = \sigma' \cap u^\perp$ (this is known as the *Separation Lemma*, [Ful93, pg. 13]). Then $\mathbb{C}[S_\sigma]_{\chi^u} \cong \mathbb{C}[S_{\sigma\cap\sigma'}] \cong \mathbb{C}[S_{\sigma'}]_{\chi^u}$.

Example 3.3. Consider the one-dimensional fan $\Delta = \{\{0\}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}\}$. Since $(\mathbb{R}_{\geq 0})^\vee$ is generated by e_1^*, so the corresponding semigroup is generated by X. Similarly $S_{\mathbb{R}_{\leq 0}}$ is generated by X^{-1}. Finally, the zero cone $\{0\}$ has semigroup $S_{\{0\}}$ generated by X and X^{-1}.

Now both $\mathbb{C}[X]$ and $\mathbb{C}[X^{-1}]$ are isomorphic - they can both be regarded as affine rings for $\mathbb{C}$. We make the identification $X \mapsto X^{-1}$. We claim that under this identification, these two copies of $\mathbb{C}$ glue to give $\mathbb{P}^1$.

Let $(z_0 : z_1)$ be homogeneous coordinates for $\mathbb{P}^1$, and set $X = z_1/z_0$. Thus $\mathbb{C}[X] \cong U_0 = \{(1 : z_1/z_0) \mid z_0 \neq 0\}$ and, since $X^{-1} = z_0/z_1$, we have that $\mathbb{C}[X^{-1}] \cong U_1$ (where U_1 is the affine open subset of $\mathbb{P}^1$ on which $z_1 \neq 0$).

To obtain $\mathbb{P}^1$ we require that, on $U_0 \cap U_1$, both $z_0 \neq 0$ and $z_1 \neq 0$. This is given in the toric variety by the isomorphism (on the intersection)$X \mapsto X^{-1}$. z_1/z_0 is identified with z_0/z_1; i.e. $(1 : z_1/z_0) = (z_0/z_1 : 1)$.

Example 3.4. Let σ_1 be the two dimensional cone generated by e_1 and e_2. Let σ_2 by the cone generated by $-e_1$ and e_2. Let Δ be the fan generated by σ_1 and σ_2. i.e. $\Delta = \{\{0\}, \pm\mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2, \sigma_1, \sigma_2\}$.

We have that $\mathbb{C}[\sigma_1] = \mathbb{C}[X, Y]$ and $\mathbb{C}[\sigma_2] = \mathbb{C}[X^{-1}, Y]$. These two rings are isomorphic to $\mathbb{C}[x, y]$, and so U_{σ_1} and U_{σ_2} are both homeomorphic to $\mathbb{C}^2$. The gluing is given by $(X, Y) \mapsto (X^{-1}, Y)$. In particular Y remains unchanged, whilst the behaviour of X is the same as in Example 3.3. Thus we deduce that $X_\Delta = \mathbb{P}^1 \times \mathbb{C}$.[5]

4. The Character Group and the 1-Parameter Subgroups

Let $T \cong (\mathbb{C}^*)^n$ be the algebraic torus of dimension n. Associated with T are two groups:

Definition 4.1. The *character group* of T is the group:

$$M := \{\chi : T \to \mathbb{C}^* \mid \chi \text{ is a morphism}\}.$$

The 1-*parameter subgroups* of T is the group:

$$N := \{\lambda : \mathbb{C}^* \to T \mid \lambda \text{ is a morphism}\}.$$

Indeed, our choice of labelling is no coincidence (see [Ful93, §2.3] for an alternative derivation):

Lemma 4.2. *$M \cong \mathbb{Z}^n$ where, for any $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, we have that $\chi^u(t_1, \ldots, t_n) = t_1^{u_1} \ldots t_n^{u_n}$, and $N \cong \mathbb{Z}^n$ where, for any $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, we have that $\lambda^v(t) = (t^{v_1}, \ldots, t^{v_n})$.*

[5]This is a specific case of a more general result. If Δ is a fan in $N_\mathbb{R}$ and Δ' is a fan in $N'_\mathbb{R}$, then the set of products $\sigma \times \sigma'$, $\sigma \in \Delta, \sigma' \in \Delta'$, forms a fan denoted by $\Delta \times \Delta'$ in $(N \oplus N')_\mathbb{R}$. At the level of varieties we have that $X_{\Delta \times \Delta'} = X_\Delta \times X_{\Delta'}$.

Proof. Any morphism $M \ni \chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$ corresponds to a ring homomorphism

$$\chi^* : \mathbb{C}[Y, Y^{-1}] \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].$$

By definition, $1 = \chi^*(YY^{-1}) = \chi^*(Y)\chi^*(Y^{-1})$, and since $\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = \mathbb{C}[X_1, \ldots, X_n]_{X_1 \ldots X_n}$ we have that there exist $G, H \in \mathbb{C}[X_1, \ldots, X_n]$ such that:

$$G \cdot H = X_1^{a_1} \ldots X_n^{a_n} \qquad \text{for some } a_1, \ldots, a_n \in \mathbb{Z}.$$

This forces G and H to be monomials. Hence $\chi^*(Y)$ (and $\chi^*(Y^{-1})$) is a monomial in $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$.

Conversely any monomial $X_1^{u_1} \ldots X_n^{u_n}$ in $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ clearly defines a ring homomorphism, sending $Y \mapsto X_1^{u_1} \ldots X_n^{u_n}$ and $Y^{-1} \mapsto X_1^{-u_1} \ldots X_n^{-u_n}$.

Similarly for $\lambda \in N$. □

Let $\chi \in M$ and $\lambda \in N$. The composition $\chi \circ \lambda : \mathbb{C}^* \to \mathbb{C}^*$ gives a map $t \mapsto t^k$ for some $k \in \mathbb{Z}$. We define $\langle \chi, \lambda \rangle := k$. In fact the map is given by:

$$\begin{aligned} \langle \cdot, \cdot \rangle : M \times N &\to \mathbb{Z} \\ (\chi^u, \lambda^v) &\mapsto u_1 v_1 + \cdots + u_n v_n, \end{aligned}$$

and is a perfect pairing (see [Ful93, pg. 37]).

Let X be a toric variety with torus $T \cong (\mathbb{C}^*)^n$. Consider some point $u \in M \cong \mathbb{Z}^n$. This corresponds to a morphism $\chi^u : T \to \mathbb{C}^*$, and since T is a dense subvariety of X, we can regard χ^u as a rational function on X.

Since X is normal, associated to this rational function is a divisor $\mathrm{div}(\chi^u)$. It is supported on the complement $X \setminus T$, which we can write as the union of a finite set of irreducible divisors, i.e.

$$X \setminus T = D_1 \cup \cdots \cup D_r.$$

Hence we can write:

$$\mathrm{div}(\chi^u) = \sum_{i=1}^{r} a_i D_i,$$

where the $a_i := \mathrm{ord}_{D_i}(\mathrm{div}(\chi^u)) \in \mathbb{Z}$ are the order of vanishing of χ^u along D_i.

In fact, for each $i = 1, \ldots, r$, there exist a unique element $v_i \in N$ such that $\langle u, v_i \rangle = a_i$. It transpires that these lattice points $v_i \in N$ generate the one-dimensional cones (or *rays*) of the fan Δ corresponding to X. See [Ful93, §3.3] for a proof of this remarkable claim. This suggests yet another method for deriving the fan associated with a toric variety.

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