Frobenius manifolds

and a one-dimensional mirror theorem

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Opening

- Roughly speaking, a Frobenius manifold is a complex manifold where over every point you have a certain algebra.
- Formally defined by Boris Dubrovin in the early 90's to describe the geometry of 2-dimensional TFTs.
- Arises naturally in very different fields such as singularity theory and enumerative geometry.
- Mirror symmetry can be phrased as an isomorphism of Frobenius manifolds.

Punchline

Main result

We find Landau-Ginzburg superpotentials (B-models) for Dubrovin-Zhang Frobenius manifolds of any Dynkin type in a uniform way.

Table of contents

What is a Frobenius manifold?

Definition: Frobenius algebra

A Frobenius algebra over $\mathbb{C}$ is a pair $((\mathcal{A}, \star), < \, , \, >)$ where:

- $(\mathcal{A}, \star)$ is an associative commutative unital $\mathbb{C}$-algebra.
- $< \, , \, >$ is a symmetric, nondegenerate $\mathbb{C}$-bilinear form such that the Frobenius property holds:

$$< a \star b, c > = < a, b \star c >, \quad \forall a, b, c \in \mathcal{A}.$$

Example (Trival)

Let $(\mathcal{A}, \star) = (\mathbb{C}, \cdot)$. Fix some $\alpha \in \mathbb{C}^*$, then let $< a, b > = \alpha ab$.

Example (Group ring)

Let G be a finite abelian group. Consider the group ring $\mathbb{C}[G]$, which has elements of the form

$$\sum_{i=1}^{|G|} a_i g_i, \qquad a_i \in \mathbb{C}, g_i \in G.$$

This is a Frobenius algebra by letting

$$< a, b > := (a \cdot b)|_{e_G}, \quad \text{for } a, b \in \mathbb{C}[G].$$

- Frobenius algebras are common and important in mathematics.

Frobenius manifold - Global definition I

A Frobenius manifold is a complex manifold M satisfying the following axioms:

1. $\forall t \in M$, T_tM is a Frobenius algebra, depending analytically on the point.
2. $< \, , \, >$ is a flat 'metric' on M.
3. If e is the unity vector field of the tangent algebra, we have
$$\nabla e = 0.$$
4. Let $u, v, w, x \in \Gamma(TM, M)$. Define a symmetric $(0-3)$-tensor c:
$$c(u, v, w) := < u \cdot v, w > .$$
Then the 4-tensor
$$(\nabla_x c)(u, v, w),$$
is symmetric in all vector fields u, v, w, x.

Frobenius manifold - Global definition II

5. Conformality: $\exists\, E \in \Gamma(TN, M)$ such that

$$\nabla(\nabla E) = 0,$$

with

$$\mathcal{L}_E(\cdot) = d(\cdot), \quad \text{and} \quad \mathcal{L}_E\eta = (2-d)\eta.$$

❁ E induces a graded structure, and is commonly called the Euler vector field.

Remark

Equivalently to bullet point 4, $u, v, w \in \Gamma(TM, M)$, there exists a function F such that

$$c(u, v, w) = u\, v\, w(F).$$

Frobenius manifold - Local definition

- Since the metric is flat: $\exists$ 'flat' coordinates, $\{t^\alpha\}$ such that $\eta_{\alpha\beta} :=< \partial_\alpha, \partial_\beta >$ are constants. Here $\partial_\alpha \equiv \dfrac{\partial}{\partial t^\alpha}$.
- Since $\nabla e = 0$ we can let $e = \partial_1$.
- From prop 4: $\exists$ a function $F(t)$ such that

$$c_{\alpha\beta\gamma} := c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial^3_{\alpha\beta\gamma} F.$$

  F is called the prepotential of M.
- The above means that

$$c_{1\alpha\beta} \equiv < \partial_1 \star \partial_\alpha, \partial_\beta > = < \partial_\alpha, \partial_\beta > = \eta_{\alpha\beta} = \partial^3_{1\alpha\beta} F.$$

- $c_{\alpha\beta}^{\gamma} \equiv c_{\alpha\beta\delta}\eta^{\delta\gamma}$ are structure coefficients for the Frobenius algebra;

$$\partial_\alpha \star \partial_\beta = c_{\alpha\beta}^{\gamma}\partial_\gamma.$$

- The Euler vector field is linear

$$E = \sum_{i|d_i\neq 0} d_i t^i \partial_i + \sum_{i|d_i=0} r_i\partial_i, \quad \text{for some } d_i, r_i \in \mathbb{C}.$$

  This means that $d_i = \deg(t^i)$, and

$$F(\lambda^{d_1}t^1, \cdots, \lambda^{d_n}t^n) = \lambda^{d_F}F,$$

  so F is a quasi-homogeneous function of degree d_F.

- By the above we have the important relation

$$\mathcal{L}_E F \equiv E(F) = d_F F.$$

The WDVV-equations!

F satisfies an (overdetermined) system of nonlinear PDEs called the WDVV equations:

$$\partial^3_{ijk} F \eta^{kl} \partial^3_{lmn} F = \partial^3_{mjk} F \eta^{kl} \partial^3_{lin} F,$$

or equivalently in terms of the c-tensor;

$$c_{ijk} \eta^{kl} c_{lmn} = c_{mjk} \eta^{kl} c_{lin}.$$

This systems governs the Frobenius manifold

Global- local correspondence

- (Global) Frobenius manifold $M \rightsquigarrow$ local structure with F satisfying WDVV.
- F solving WDVV $\rightsquigarrow$ local structure $\rightsquigarrow$ (Global) Frobenius manifold.

Caveat: WDVV are nonlinear PDEs - hard to find solutions.

Example (Trivial)

$$M = \mathbb{C}, \quad e = \partial_t, \quad \eta =< e, e >= 1, \quad E = t\,\partial_t \implies F = \frac{t^3}{3!}$$

Example (Cohomology of the complex projective line)

$$M = H(\mathbb{P}^1, \mathbb{C}) \cong \mathbb{C}^2, \quad F = \frac{t_1^2 t_2}{2} + e^{t_2}, \quad e = \partial_1, \quad E = t_1\partial_1 + 2\partial_2,$$

$$\partial_2 \star \partial_2 = e^{t_2}\partial_1 \implies (\mathcal{A}, \star) \cong \mathbb{C}[x]/(x^2 - e^{t_2})$$

How can I make one?

3 main ways

1. Quantum cohomology (A-model)
2. Singularity theory (B-model)
3. Lie theory (C-model??)

1) Quantum cohomology - A-model

- $M = QH^\bullet(X)$, for example for X smooth projective complex algebraic variety.
- A deformation of the usual cohomology.
- η: Poincaré pairing, E: de Rham grading.
- Important to take the evenly graded part.
- F is a generating function for the genus 0 Gromov-Witten invariants of X.

2) Singularity theory/Landau-Ginzburg superpotential - B-model

- A family of holomorphic functions $\lambda(p, \underline{u}) : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$, such that λ is morse for a generic point in $\mathbb{C}^n$.
- $M = \mathbb{C}^n, \quad T_{\underline{u}}M = \mathbb{C}[u_1, \cdots, u_n]/\left(\frac{\partial \lambda}{\partial p}\right)$.
- Critical values u_i of λ may serve as local (canonical) coordinates.
- In these coordinates:

$$\eta_{ij} = \frac{\delta_{ij}}{\lambda''}, \quad \partial_i \star \partial_j = \delta_{ij}\partial_i, \quad E = \sum_{i=1}^{n} u_i \partial_i$$

- In general: Use Saito theory to find flat coordinates and η, c by taking residues.

Example

The miniversal deformation of a type A_2 singularity:

Example

$$p^2 \mapsto p^3 + a_2 p + a_1 = \lambda(p, \underline{a}).$$

Natural coordinates: $\{a_1, a_2\}$, and for $e = \partial_1$:

$$\eta = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad E = t_1 \partial_1 + \frac{2t_2}{3} \partial_2, \quad \partial_2 \cdot \partial_2 = -\frac{t_2}{9} \partial_1, \implies F = \frac{t_1^2 t_2}{6} - \frac{t_2^4}{216}.$$

(A/B) Mirror symmetry

A	B
Frobenius manifold from $QH^\bullet(X)$	Frobenius manifold from $\lambda(p, \underline{u})$

Example

Quantum cohomology of $\mathbb{P}^1$	$\lambda(\mu, \underline{u}) = \dfrac{u_0(1 + \mu^2 + u_1\mu)}{\mu}$

With $\mu = e^p$.

Lie theory (C-model?)

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ be the Cartan subalgebra, $\mathcal{W}_{\mathfrak{g}}$ the corresponding Weyl group. Then

- $M = \mathfrak{h}^{\text{reg}}/\mathcal{W}_{\mathfrak{g}}$: the space of regular orbits.
- A flat metric on M, g, is induced by the Cartan-Killing form.
- In flat coordinates: $\eta_{\alpha\beta} := \partial_1 g_{\alpha\beta}$.
- $F(t)$ is a polynomial.

Examples

Example

$\mathfrak{g} = \mathfrak{sl}_2$, $\rightsquigarrow$ the trivial Frobenius manifold as before with $F = \dfrac{t^3}{6}$.

Example

$\mathfrak{g} = \mathfrak{sl}_3, \quad F(t) = \frac{t_1^2 t_2}{6} - \frac{t_2^4}{216}, \quad E = t_1\partial_1 + \frac{2t_2}{3}\partial_2.$

B/C mirror symmetry

B	C
Frobenius manifold from $\lambda(p, \underline{u})$	Frobenius manifold on $M = \mathfrak{h}/\mathcal{W}_{\mathfrak{g}}$

Example

$\lambda(p, \underline{a}) = p^3 + a_2 p + a_1$	$\mathfrak{g} = \mathfrak{sl}_2$

Dubrovin-Zhang Frobenius manifolds

DZ-manifolds - a C-model

Take the affine version of the Lie theory construction before with a 'twist'.

- Take the affine situation, but make an extension:

$$\hat{\mathcal{W}}_{\mathfrak{g}}^{k} = \mathcal{W}_{\mathfrak{g}} \rtimes \Lambda^{\vee} \rtimes \mathbb{Z}$$

acts on $\mathfrak{h} \times \mathbb{C}$.

- Now g would be induced by an orthogonal extension of the Cartan-Killing form.

- Depends on a special 'canonical' Dynkin node k.

Theorem (B.Dubrovin, Y. Zhang '98)

For any Dynkin type $\exists!$ *semi-simple Frobenius manifold structure on* $M_{\mathfrak{g}}^k = (\mathfrak{h} \times \mathbb{C})/\hat{\mathcal{W}}_{\mathfrak{g}}^k$ *such that*

- $e = \partial_k$,
- $E = \sum_{i=1}^{rk} \left(\frac{d_i}{d_k} t_i \partial_i\right) + \frac{1}{d_k} \partial_{rk+1}$, *where* $d_i = (\omega_i, \omega_k)$,
- $F \in \mathbb{C}[t_1, \cdots, t_{rk}][e^{t_{rk+1}}]$.

- Issues with this approach: no explicit transformation into flat coordinates. Don't get F explicitly, computer limitations!

Example + mirror symmetry

Example

$$\mathfrak{g} = \mathfrak{sl}_2, \quad F = \frac{t_1^2 t_2}{2} + e^{t_2}, \quad e = \partial_1, \quad E = t_1\partial_1 + 2\partial_2.$$

Example

B	C
$\lambda(\mu, \underline{u}) = u_0(1 + \mu^2 + u_1\mu)/\mu$	$M^1_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sl}_2$

With $\mu = e^p$.

B-model attempts

❁ In [1]: Dubrovin and Zhang: superpotential for affine type A_n ($\mathfrak{sl}_{n+1}$).

❁ In [2]: the authors of [1] with Strachan, Zuo: superpotential for all remaining affine classical cases (B_n, C_n, D_n / $\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n},$).

❁ In [3]: Brini, using a completely different strategy to the above: superpotential for the affine E_8.

The theorem

A unified B-model construction

✾ Fix nontrivial $\rho \in \mathrm{Rep}(\exp(\mathfrak{g}))$,

Motivated by physics (for simply laced): considered the characteristic equation for the Lax operator of a ‘relativistic’ Toda lattice;

$$\begin{aligned}
&\det\left(\hat{L}_{\mathfrak{g}}(\lambda) - \mu\mathbb{1}\right) = 0 \\
&= \sum_{i=0}^{\dim(\rho)} (-1)^i \mathrm{Tr}(\wedge^i \rho)\mu^i |_{u_k \mapsto u_k + \lambda/u_0} = 0 \\
&= \sum_{i=0}^{\dim(\rho)} (-1)^i \mathfrak{p}(u_0, \cdots, u_{\mathrm{rk}}; \lambda)\mu^i = 0.
\end{aligned}$$

The result

Theorem (A.Brini '17, A.Brini-KvG '21)

$$M^k_{\lambda_{\mathfrak{g}}} \cong M^k_{\mathfrak{g}}, \quad \forall \mathfrak{g}.$$

Remark

- Recovers the previous B-model constructions.
- Is solvable by computer and in relatively short time.
- Obtain F explicitly.

How to prove

✾ For simplicity we chose the minimal nontrivial irreducible representation (but results in integrable systems implies the choice doesn't matter).

1. Find the necessary relations in the representation ring to obtain the curve.
2. In general, consider the Laurent expansions of the curve (in canonical coordinates) at every singular point
3. and use Hurwitz theory to find flat coordinates and to calculate c_{0ij}.
4. Then WDVV fixes all of c by solving for the coefficients of F (by using the polynomial ansatz constrained by the Euler vector field and the uniqueness from the DZ-theorem).

Example: G_2

Here $\dim(\rho) = 7$, $k = 2$, and we get the relations:

$$\mathfrak{p}_i = \mathfrak{p}_{7-i}, \quad \mathfrak{p}_0 = 1, \quad \mathfrak{p}_1 = u_1, \quad \mathfrak{p}_2 = u_1 + u_2 + \frac{\lambda}{u_0}, \quad \mathfrak{p}_3 = u_1^2 - u_2 - \frac{\lambda}{u_0},$$

which gives

$$\lambda = -\frac{u_0\left(\mu^6 + (1-u_1)\mu^5 + (1+u_2)\mu^4 + (1-u_1^2+2u_2)\mu^3 + (1+u_2)\mu^2 + (1-u_1)\mu + 1\right)}{\mu^2(\mu+1)^2},$$

which (after nontrivial calculations) gives

$$F = \frac{1}{2}t_2^2 t_0 + \frac{1}{4}t_2 t_1^2 - \frac{1}{96}t_1^4 + \frac{1}{3}t_1^3 e^{3t_0} + \frac{1}{2}t_1^2 e^{6t_0} + \frac{1}{12}e^{12t_0}.$$

Applications

Applications

- Topological degree of LL-maps.
- Seiberg-Witten theory.
- Integrable hierarchies.
- Gromov-Witten theory of orbicurves:

Conjecture

Applying Topological recursion to $\lambda_{A,D,E}$ should recover the (stationary, orbifold) Gromov-Witten theory of $\mathbb{P}^1$ with 2/3 orbifold points of certain weights.

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Thanks!