Examples of Generalized Complex Structures

Gil Cavalcanti

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Complex Structures
the tangent space point of view

A complex structure in M^{2n} is

- A distribution $L \subset T_{\mathbb{C}}M$;
- $\dim_{\mathbb{C}} L = n$;
- L is closed under $[\cdot,\cdot]$ (integrable);
- $L \cap \overline{L} = \{0\}$ (nondegenerate).

L defines the $(1,0)$-vectors.

Complex Structures
the differential forms point of view

A complex structure in M^{2n} is

- $\Omega = \theta_1 \dots \theta_n$ locally decomposable n-form;
- $d\Omega = \alpha\Omega$ (integrable);
- $\Omega \wedge \overline{\Omega} \neq 0$ (nondegenerate).

Ω is a local $(0, n)$ form.

L is the annihilator of Ω.

Generalized Complex Structure
Ingredients

- Clifford action of $T \oplus T^*$ on forms;

- Courant bracket:

  $[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}(d(X\lfloor\eta - Y\lfloor\xi))$;

- The natural pairing on $T \oplus T^*$:

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y));$$

- The Mukai pairing on forms:

$$\langle\rangle.$$

Geneneralized Complex Structures
the $T \oplus T^*$ point of view

A generalized complex structure in M^{2n} is

- A distribution $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$;
- $\dim_{\mathbb{C}} L = 2n$;
- L is isotropic wrt the natural pairing;
- L is closed under $[\cdot, \cdot]$ (integrable);
- $L \cap \overline{L} = \{0\}$ (nondegenerate).

Generalized Complex Structures
the differential forms point of view

A geneneralized complex structure in M^{2n} is

- $\rho = \Omega \exp(B+i\omega)$, Ω locally decomposable;

- $d\rho = (X+\xi)\cdot\rho$ (integrable);

- $\langle \rho, \overline{\rho} \rangle = \Omega \wedge \overline{\Omega}\omega^k \neq 0$ (nondegenerate).

A generalized structure ρ gives us the *spinor* line bundle.

L is the annihilator of ρ.

The Jacobi identity

- The Courant bracket does not satisfy the Jacoby identity.

- $Jac(X, Y, Z) = d(\langle [X, Y], Z\rangle + \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle)$.

- The Courant bracket satisfies the Jacobi identity in L.

- Use $[\cdot, \cdot]$ to define $d : \bigwedge^* L^* \to \bigwedge^{*+1} L^*$.

The B-field

- A 2-form B acts on $T \oplus T^*$ by

$$X + \xi \mapsto X + \xi - X \lfloor B;$$

- B skew-symmetric $\Rightarrow$ B is orthogonal;

- B closed $\Rightarrow$ B preserves $[\cdot, \cdot]$;

- $L \mapsto L^B = \{X + \xi - X \lfloor B | X + \xi \in L\}$;

- $\rho \mapsto \rho \exp(B)$.

Examples

- Complex Structures:

$$\rho = \Omega; \qquad L = T^{1,0} \oplus T^{*0,1};$$

- Symplectic Structures:

$$\rho = \exp(i\omega); \quad L = \{X - iX\lfloor\omega \mid X \in T_{\mathbb{C}}M\};$$

- Products and B-field transform.

Local structure

Around a *regular point* the structure gives a foliation of the manifold with with symplectic leaves and complex base.

Idea

Symplectic fibration + generalized complex base + Thurston's argument:

Theorem

If a symplectic fibration over a generalized complex base is such that the base and the fibers are 1–connected, then the total space has a generalized complex structure.

More examples

Principal torus bundles E over surfaces always have generalized complex structures:

- Take the complex structure on the base Ω
- Let $\rho = \Omega \exp(id\theta_1 d\theta_2)$;
- $b_1(E)$ even $\Rightarrow$ no complex structure (Kodaira);
- Euler class $(m, n) \neq 0$ + genus $> 1 \Rightarrow$ no symplectic structure (Walczak & Etgü)

Nilmanifolds

- Quotient of nilpotent Lie group by a maximal rank lattice;
- "Iterated circle bundles over a point";
- There are 34 of those in 6-d;
- Classification of complex structures in 6-d (Salamon);
- Classification of symplectic structures in 6-d (?).

Nilpotent Lie algebras

- For $g_1 = g$ and $g_i = [g_{i-1}, g]$,

$$g_1 \supset g_2 \supset \cdots \supset g_i \supset g_{i+1} = 0$$

- Dualizing

$$d(g_k^*) \subset \bigwedge^2 (g_{k-1}^*)$$

- Presentation:

$$(0, 0, 0, 12, 13, 14)$$

$$de_1 = de_2 = de_3 = 0$$

$$de_4 = e_{12}, de_5 = e_{13}, de_6 = e_{14}$$

- Reading the brackets

$$[x_1, x_2] = -x_4, [x_1, x_3] = -x_5, etc$$

Invariant gen. complex structures

Theorem 1 *Integrability + invariance $\Rightarrow$ the spinor is closed (generalized Calabi-Yau)*

- $\rho = \theta_1 \dots \theta_k \exp(B + i\omega), \quad d\rho = (X + \xi)\rho$ implies

- $(\theta_1 \dots \theta_{k-1} d\theta_i)\theta_k = 0$

- Nilpotency gives

  $$\theta_1 \dots \theta_{i-1} d\theta_i = 0$$

  therefore

  $$d(\theta_1 \dots \theta_k) = 0;$$

- and $d\rho = 0$.

Using nilmanifold structure to find obstructions

- If $\frac{g_i}{g_{i+1}}$ is 1-d for $i \leq j \Rightarrow$ No gcs of type $(k, n-k)$, for $k \geq 2n - nil(M) + j + 1$

- maximal nilpotency index $\Rightarrow$ no gcs of type $(k, n-k)$ for $k \geq 2$.

Deformations of complex structures
— the pedestrian way —

- Complex structure: $\Omega = \theta_1 \theta_2 \dots \theta_n$;

- Then $\theta_1 \dots \theta_{n-2}$ is closed;

- And
$$\rho_t = t\theta_1 \dots \theta_{n-2}\, \mathrm{exp}(\frac{\theta_{n-1}\theta_{n-2}}{t})$$
interpolates between an $(n,0)$ and an $(n-2,2)$ structure

- In 6-d, complex can always be deformed into a general type structure

Deformations of complex structures
— revised —

- Deformations $\Leftrightarrow \beta \in \bigwedge^2 L^*$ such that

$$d_L\beta + \frac{1}{2}[\beta, \beta] = 0.$$

- Complex case:

$$\beta \in \bigwedge^2 T^{0,1}M \text{ and } [\beta, \beta] = 0$$

- Nilmanifolds with complex structure $\theta_1 \dots \theta_n$.

$$X_{n-1}, X_n \text{ duals to } \theta_{n-1}, \theta_n;$$

- $\beta = X_{n-1} \wedge X_n$ defines a deformation of gcss.

Iwasawa manifold

$(0, 0, 0, 0, 13 - 24, 14 + 23)$

- The space of complex structures has 2 components (‘Good George’ and Salamon);

- Complex structures in different components determine different orientations on the base 4-torus;

- $$\rho_1 = (e_1 + ie_2)(e_3 + ie_4)(e_5 + ie_6)$$

  and

  $$\rho_2 = (e_1 + ie_2)(e_3 - ie_4)(e_5 - ie_6)$$

  are in distinct components;

- ρ_1 can be deformed by $\beta = \frac{-1}{4}(x_3 - ix_4)(x_5 - ix_6)$ to

$$(e_1 + ie_2)\,\mathsf{exp}(-(e_{35} - e_{46}) - i(e_{45} + e_{36}))$$

- Similarly, ρ_2 can be deformed to

$$(e_1 + ie_2)\,\mathsf{exp}(e_{35} - e_{46} - i(e_{45} + e_{36}))$$

- Both are B-field transforms of

$$(e_1 + ie_2)\,\mathsf{exp}(-i(e_{45} + e_{36}))$$

- Conclusion: *Space of complex structures on the Iwasawa manifold can be connected using gcss*

An 8-d example

Consider the nilmanifold

$$(0, 0, 12, 13, 14, 15, 16, 36 - 45 - 27)$$

- Maximal nilpotence step $\Rightarrow$ no gcs of type $(4, 0), (3, 1)$ or $(2, 2)$.

- A $(1, 3)$ structure would imply symplectic structure on the leaves:

  $$(0, 0, 0, 0, 0, 14 - 23)$$

  and there aren't any!

- A $(0, 4)$ structure is just symplectic structure, but

  $$H^2(M) = \text{span}\{e_{23}, e_{34} - e_{25}, e_{17}\}.$$

  There is no e_8 above $\Rightarrow$ no symplectic form.

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